Mathematical Details on the
Regensburg Formula

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Three theorems and their proofs concerning the Regensburg Formula

Let

- E_{BY}^{i} be the national emissions for each country *i* in a base year (*BY),*
- $\{E_t\}$ be any global emission pathway consisting of the annual global emissions E_{BY+1}, E_{BY+2}, …, E_{CY}, with the global convergence amount E_{CY} being reached for the first time in the convergence year *CY*, as well as
- E_{CY}^{i} be the national convergence amount for each country *i* in the convergence year *CY.*

E_{BY} denotes the global emissions in the base year.

Therefore

$$\sum_i E_{BY}^i = E_{BY},$$

$$E_t \neq E_{CY} \text{ for } BY \leq t \leq CY - 1,$$

$$\sum_i E_{CY}^i = E_{CY}.$$

E_t^i denotes the emissions of country *i* in year *t*. The period from the beginning of the year *BY+1* until the end of the year *CY* is denominated convergence period.

The emissions of country *i* in the convergence period are denominated national budget of country *i* in the convergence period:

$$BG^i := \sum_{t=BY+1}^{CY} E_t^i.$$

The global emissions in the convergence period are denominated global budget in the convergence period:

$$BG := \sum_{t=BY+1}^{CY} E_t \left(= \sum_{t=BY+1}^{CY} \sum_i E_t^i = \sum_i \sum_{t=BY+1}^{CY} E_t^i = \sum_i BG^i \right).$$

With the help of the Regensburg Formula, departing from any arbitrary global emission pathway $\{E_t\}$ which reaches a given global convergence amount E_{CY} for the first time in the convergence year *CY*, a national emission pathway $\{E_t^i\}$ can be calculated for each country *i* which reaches in the convergence year *CY* a given national convergence amount E_{CY}^i.
If the global emission pathway gradually approximates the global convergence amount E_{CY}, this gradual approximation is transmitted to all national emission pathways.

Below we present three equivalent notations of the Regensburg formula

- as a weight function with an annual degree of of achieving the global convergence amount
- as a straight line with a conversion factor for the reduction of emissions
- as a recursion with an annual rate of change

and show how they are derived from each other. Then we show the derivation of a formula for the national budget in the convergernce periode of an individual country.

The notation of the **Regensburg Formula as a weight function** uses the annual degree of achieving the global convergence amount E_{CY} in year *t*

$$\boldsymbol{C_t := \frac{E_{BY} - E_t}{E_{BY} - E_{CY}}}$$

as weighting factor for the the national convergence amount E_{CY}^i (in case of the national convergence amount being directly proportional to the population, it is also per-capita weighting factor) for the calculation of emissions of country *i* in year *t*:

$$\boldsymbol{E_t^i := (1 - C_t) * E_{BY}^i + C_t * E_{CY}^i}, \ BY + 1 \leq t \leq CY$$

Directly from this definition of the Regensburg Formula we obtain the following results:

Remark 1 (equal proportions in all countries and the world)
In each year *t*, the proportion of emissions still to be reduced and the proportion of emissions already reduced in relation to the emissions to be reduced altogether are equal in all countries and globally:

$$\frac{E_t - E_{CY}}{E_{BY} - E_{CY}} = \frac{E_t^i - E_{CY}^i}{E_{BY}^i - E_{CY}^i} \; (= 1 - C_t) \text{ and}$$

$$\frac{E_{BY} - E_t}{E_{BY} - E_{CY}} = \frac{E_{BY}^i - E_t^i}{E_{BY}^i - E_{CY}^i} \; (= C_t).$$

In each year *t*, therefore, the degree of achieving the global convergence amount and the degree of achieving the national convergence amount are identical.

Remark 2 (national convergence amounts in all countries in the convergence year)
In the convergence year *CY* emissions calculated with the Regensburg Formula and the national convergence amount are the same in each country.

Theorem 1 (notation of the Regensburg Formula as a straight line)
We have:

$$\boldsymbol{E_t^i = (E_t - E_{CY}) * a^i + E_{CY}^i}, \quad BY + 1 \le t \le CY,$$

with the conversion factor for the reduction: $\boldsymbol{a^i} := \frac{\boldsymbol{E_{BY}^i - E_{CY}^i}}{\boldsymbol{E_{BY} - E_{CY}}}$.

Proof:

$$E_t^i =$$

$$= E_{BY}^i * (1 - C_t) + C_t * E_{CY}^i =$$

$$= E_{BY}^i * \left(1 - \frac{E_{BY} - E_t}{E_{BY} - E_{CY}}\right) + \left(\frac{E_{BY} - E_t}{E_{BY} - E_{CY}}\right) * E_{CY}^i =$$

$$= E_{BY}^i * \left(\frac{E_t - E_{CY}}{E_{BY} - E_{CY}}\right) + \left(1 - \frac{E_t - E_{CY}}{E_{BY} - E_{CY}}\right) * E_{CY}^i =$$

$$= (E_t - E_{CY}) * \frac{E_{BY}^i - E_{CY}^i}{E_{BY} - E_{CY}} + E_{CY}^i =$$

$$= (E_t - E_{CY}) * a^i + E_{CY}^i$$

□

Remark 3 (stepwise approximation)

By presenting the Regensburg Formula as a straight line, it becomes clear that a stepwise approximation of the global emission pathway to the global convergence amount is transmitted to all national emission pathways.

Remark 4 (construction of national graphs)

This theorem also shows that, when applying the Regensburg Formula, the national graph (t, E_t^i) for country i with a reduction amount $(E_{BY}^i > E_{CY}^i)$ can be derived from the global graph (t, E_t) by changing the scaling on the ordinate and by vertically shifting the abscissa. For countries with a national convergence amount permitting increasing annual emissions $(E_{BY}^i < E_{CY}^i)$, the global graph additionally needs to be reflected across the abscissa to obtain the national graph.

Remark 5 (factor for converting reductions = proportional factor)

Because of $\sum_i a^i = 1$ the factor for converting the reduction is also called “proportional factor”.

Corollary 1 (constant factor for converting reductions)

For each country *i* there is a constant proportional factor α^i that allows converting annual global reductions to annual reductions of country *i*:

$$E_t^i - E_{t-1}^i = (E_t - E_{t-1}) * a^i.$$

Factor a^i for converting reductions can be determined by the ratio between emissions that remain to be reduced by country *i* in year *t* and emissions which remain to be reduced globally:

$$a^i = \frac{E_t^i - E_{CY}^i}{E_t - E_{CY}} \quad (BY \leq t \leq CY - 1).$$

Remark 6 (**monotonicity**)

This corollary also shows that monotonicity of the global emission pathway is transferred to the national emission pathways.

Corollary 2 (complete distribution of global emissions)

The emissions determined according to the Regensburg Formula of all countries together sum up to the amount of global emissions:

$$\sum_i E_t^i = E_t \quad \text{for every year } t.$$

Proof by using the notation of the Regensburg Formula as a straight line:

$$\sum_i E_t^i =$$

$$= \sum_i \left((E_t - E_{CY}) * a^i + E_{CY}^i \right) =$$

$$= (E_t - E_{CY}) * \sum_i a^i + \sum_i E_{CY}^i =$$

$$= (E_t - E_{CY}) * 1 + E_{CY} =$$

$$E_t$$

□

Theorem 2 (notation of the Regensburg Formula as a recursion)

We have:[1]

$$\boldsymbol{E_t^i = E_{t-1}^i - CR_{t-1} * (E_{t-1}^i - E_{CY}^i)}, \qquad BY + 1 \le t \le CY$$

with the annual rate of change $\boldsymbol{CR_{t-1}} := \dfrac{\boldsymbol{E_{t-1} - E_t}}{\boldsymbol{E_{t-1} - E_{CY}}}$.

Proof:

CR_{t-1} is well defined, because $E_{t-1} \neq E_{CY}$ for $BY + 1 \le t \le CY$.

By using corollary 1 for the factor for converting reductions, we can say:

$$E_t^i =$$

$$= E_{t-1}^i + (E_t - E_{t-1}) * a^i =$$

$$= E_{t-1}^i - \frac{E_{t-1} - E_t}{E_{t-1} - E_{CY}} * \left(E_{t-1}^i - E_{CY}^i \right) =$$

$$= E_{t-1}^i - CR_{t-1} * (E_{t-1}^i - E_{CY}^i)$$

□

[1]: Alternative notation with $TA := E_{CY}, TA^i := E_{CY}^i$ and $\widetilde{CR}_{t-1} := -CR_{t-t} = \frac{E_t - E_{t-1}}{E_{t-1} - TA}$:

$$E_t^i = E_{t-1}^i + \widetilde{CR}_{t-1} * (E_{t-1}^i - TA^i).$$

Remark 7 (identical annual rates of change)

The notation as a recursion offers another interpretation of the Regensburg Formula: The annual emissions of country *i* in year *t* are determined by transferring the rates of change which are derived from the global emission pathway, to national emission pathways. Therefore, in each year *t*, the national and global annual rates of change are identical.

Remark 8 (national convergence amounts in all countries in the convergence year)

From the notation of the Regensburg Formula as a recursion, you can see that the convergence amounts are achieved in all countries in year *CY*, if you take into consideration that the rate of change CR_{CY-1} takes on value 1.

Theorem 3 (national budget)

For the national budget of country *i* in the convergence period we have:

$$BG^i = E^i_{CY} * (CY - BY) + (BG - E_{CY} * (CY - BY)) * \ a^i,$$

$$\text{with the factor } a^i = \frac{E^i_{BY} - E^i_{CY}}{E_{BY} - E_{CY}} \text{ for converting reductions.}$$

Proof:

According to the notation of the Regensburg Formula as a straight line, the following applies to the emissions of country *i* in year *t*:

$$E^i_t = (E_t - E_{CY}) * a^i + E^i_{CY}.$$

By summing up these emissions across all years, we obtain the national budget of country *i* in the convergence period:

$$BG^i = \sum_{t=BY+1}^{CY} E^i_t =$$

$$= \sum_{t=BY+1}^{CY} E^i_{CY} + \sum_{t=BY+1}^{CY} (E_t - E_{CY}) * a^i$$

$$= E^i_{CY} * (CY - BY) + (BG - E_{CY} * (CY - BY)) * \ a^i$$

□

Remark 9 (national budget depending only on the global budget)

This theorem also shows, that the national budget of country *i* in the convergence period only depends on – besides the national emissions of country *i* and the global emissions in the base year and in the convergence year – the global budget in the convergence period, but not on the global emissions E_{BY+1}, E_{BY+2}, …, E_{CY-2}, E_{CY-1}.